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Existence of three positive solutions of m-point boundary value problems for some nonlinear fractional differential equations on an infinite interval

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ABSTRACT

In this paper, we investigate the existence of three positive solutions for the following m-point fractional boundary value problem on an infinite interval

$$D^{\alpha}_{0+}u(t) + a(t)f(u(t)) = 0, \quad 0 < t < +\infty,$$

$$u(0) = u'(0) = 0, \qquad D^{\alpha-1}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i),$$

where $2 < \alpha < 3$, D^{α}_{0+} is the standard Riemann–Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty$, $\beta_i \geq 0$, $i = 1, 2, \ldots, m-2$ satisfies $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$. The method involves applications of a fixed point theorem due to Leggett–Williams. As applications, examples are presented to illustrate the main results.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical models of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so on, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. For an extensive collection of such results, we refer the readers to the monographs by Samko et al. [1], Podlubny [2] and Kilbas et al. [3]. For the basic theory and recent development of the subject, we refer a text by Lakshmikantham [4]. For more details and examples, see [5–15] and the references therein. However, the theory of boundary value problems for nonlinear fractional differential equations is still in the initial stages and many aspects of this theory need to be explored. Especially, the existence of positive solutions for m-point fractional boundary value problem on infinite interval are relatively scarce.

Li et al. [12] considered the following three point boundary value problems of the fractional order differential equation

$$\begin{aligned} &D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1,\ 1 < \alpha \leq 2, \\ &u(0) = 0, \qquad D^{\beta}_{0_+}u(1) = aD^{\beta}_{0_+}u(\xi), \end{aligned}$$

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where D^{α}_{0+} is the standard Riemann–Liouville fractional derivative. The existence and multiplicity results of positive solutions are obtained by using some fixed-point theorems.

Zhao and Ge [16] considered the following fractional order differential equation

$$\begin{aligned}&D^{\alpha}_{0+}u(t)+f(t,u(t))=0, \quad 0<t<+\infty,\ 1<\alpha\le 2,\\ &u(0)=0, \qquad \lim_{t\to\infty} D^{\alpha-1}_{0_+}u(t)=\beta u(\xi),\end{aligned}$$

where D^{α}_{0+} is the standard Riemann–Liouville fractional derivative. The existence and multiplicity results of positive solutions are obtained by using some fixed-point theorems.

Motivated by papers [12,16], in this paper, we deal with the following nonlinear m-point fractional boundary value problem on an infinite interval

$$D^{\alpha}_{0+}u(t)+a(t)f(u(t))=0, \quad 0<t<+\infty, \tag{1.1}$$

$$u(0)=u'(0)=0, \qquad D^{\alpha-1}u(+\infty)=\sum_{i=1}^{m-2}\beta_i u(\xi_i), \tag{1.2}$$

where $2<\alpha<3$, D^{α}_{0+} is the standard Riemann–Liouville fractional derivative, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<+\infty$, $\beta_i\ge 0$, $i=1,2,\ldots,m-2$ satisfies $0<\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}<\Gamma(\alpha)$.

To the authors' knowledge, no one has studied the existence of positive solutions for fractional boundary value problems (1.1) and (1.2). The goal of present paper is to use the fixed-point index theory due to Leggett–Williams, we obtain sufficient conditions for the existence of three positive solutions for the following m-point fractional boundary value problem on infinite interval (1.1) and (1.2).

Throughout this paper, we assume that the following conditions hold:

(C_1) $f\in C([0,+\infty),[0,+\infty))$, $f(u)\not\equiv 0$ on any subinterval of $(0,+\infty)$ and when u is bounded $f((1+t^{\alpha-1})u)$ is bounded on $[0,+\infty)$;

(C_2) $a:[0,+\infty)\to[0,+\infty)$ is not identical zero on any closed subinterval of $[0,+\infty)$, and

$$0<\int_0^{+\infty}a(s)\mathrm{d}s<\infty.$$

2. Preliminaries

We need the following definitions and lemmas that will be used to prove our the main results.

Definition 2.1 (*[2]*)**.** The integral

$$I^{s}_{0+}f(x)=\frac{1}{\Gamma(s)}\int_0^x\frac{f(t)}{(x-t)^{1-s}}\mathrm{d}t, \quad x>0,$$

where $s>0$, is called the Riemann–Liouville fractional integral of order s and $\Gamma(s)$ is the Euler gamma function defined by

$$\Gamma(s)=\int_0^{+\infty}t^{s-1}\mathrm{e}^{-t}\mathrm{d}t, \quad s>0.$$

Definition 2.2 (*[3]*)**.** For a function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D^{s}_{0+}f(x)=\frac{1}{\Gamma(n-s)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n}\int_0^x\frac{f(t)}{(x-t)^{s-n+1}}\mathrm{d}t,$$

where $n=[s]+1$, $[s]$ denotes the integer part of number s, is called the Riemann–Liouville fractional derivative of order s.

The following two lemmas can be found in [3,7] which are crucial in finding an integral representation of fractional boundary value problem (1.1) and (1.2).

Lemma 2.1 (*[3,7]*)**.** *Let $\alpha>0$ and $u\in C(0,1)\cap L(0,1)$. Then the fractional differential equation*

$$D^{\alpha}_{0+}u(t)=0$$

has

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+\cdots+c_nt^{\alpha-n}, \quad c_i\in\mathbb{R},\ i=0,1,\ldots,n,\ n=[\alpha]+1$$

as a unique solution.

Lemma 2.2 ([3,7]). *Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, \ldots, n$, $n = [\alpha] + 1$.

In this section, we provide some background definitions cited from cone theory in Banach spaces.

Definition 2.3. Let $(E, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following are satisfied:

(a) if $y \in P$ and $\lambda \geq 0$, then $\lambda y \in P$;
(b) if $y \in P$ and $-y \in P$, then $y = 0$.

If $P \subset E$ is a cone, we denote the order induced by P on E by $\leq$, that is, $x \leq y$ if and only if $y - x \in P$.

Definition 2.4. A map α is said to be a nonnegative, continuous, concave functional on a cone P of a real Banach space E, if

$$\alpha : P \to [0, \infty)$$

is continuous, and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Definition 2.5. Given a nonnegative continuous functional γ on a cone P of E, for each $d > 0$ we define the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

The following fixed point theorem is fundamental and important to the proofs of our main results.

Theorem 2.1 ([17]). *Suppose that* P *is a cone of Banach space,* γ *is a nonnegative continuous concave functional of* P, *for positive real numbers* a, b, c, *we define the following set*

$$P_c = \{x \in P : \|x\| < c\},$$
$$P(\gamma, a, b) = \{x \in P : a \leq \gamma(x), \|x\| \leq b\}.$$

If $x \in \overline{P}_c$, *we have* $\gamma(x) \leq \|x\|$, $T : \overline{P}_c \to \overline{P}_c$ *is a completely continuous operator, there exists a constant* d *which satisfies* $0 < a < b < d \leq c$, *where the following conditions hold:*

(H_1) $\{x \in P(\gamma, b, d) : \gamma(x) > b\} \neq \varnothing$, *and when* $x \in P(\gamma, b, d)$, *we have* $\gamma(Tx) > b$;
(H_2) *when* $x \in \overline{P}_a$, *we have* $\|Tx\| < a$;
(H_3) *when* $x \in P(\gamma, b, c)$, $\|Tx\| > d$, *we have* $\gamma(Tx) > b$.

Then T *has at least three fixed points* x_1, x_2, x_3, *such that*

$$\|x_1\| < a, \qquad \gamma(x_2) > b, \qquad \|x_3\| > a \quad \text{and} \quad \gamma(x_3) < b,$$

specially, if $d = c$, *then condition* (H_3) *can be obtained from condition* (H_1).

3. Related lemmas

To prove the main results in this paper, we will employ several lemmas.

Lemma 3.1. *Let* $h \in C[0, +\infty)$, *then the fractional boundary value problem*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < +\infty,\ 2 < \alpha < 3, \tag{3.1}$$

$$u(0) = u'(0) = 0, \qquad D^{\alpha-1} u(+\infty) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i) \tag{3.2}$$

has a unique solution

$$u(t) = \int_0^{\infty} G(t, s) h(s) \mathrm{d}s, \tag{3.3}$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s) \tag{3.4}$$

and

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}t^{\alpha-1}-(t-s)^{\alpha-1}, & 0\le s\le t<+\infty,\\ t^{\alpha-1}, & 0\le t\le s<+\infty.\end{cases} \tag{3.5}$$

$$G_2(t,s)=\frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}G_1(\xi_i,s). \tag{3.6}$$

Proof. By Lemma 2.2, the solution of (3.1) can be written as

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2}+c_3t^{\alpha-3}-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s.$$

From $u(0)=u'(0)=0$, we know that $c_2=c_3=0$.

On the other hand, together with $D^{\alpha-1}u(+\infty)=\sum_{i=1}^{m-2}\beta_i u'(\xi_i)$, we have

$$c_1=\frac{1}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\left[\int_0^{+\infty}h(s)\mathrm{d}s-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\right].$$

Therefore, the unique solution of the fractional boundary value problem (3.1)–(3.2) is

$$\begin{aligned}u(t)&=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s+\frac{t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\int_0^{+\infty}h(s)\mathrm{d}s\\
&\quad-\frac{t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\
&=\int_0^{+\infty}G_1(t,s)h(s)\mathrm{d}s+\frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\int_0^{+\infty}\frac{\xi_i^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s-\frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\
&=\int_0^{+\infty}G_1(t,s)h(s)\mathrm{d}s+\frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}\int_0^{+\infty}G_1(\xi_i,s)h(s)\mathrm{d}s\\
&=\int_0^{+\infty}G_1(t,s)h(s)\mathrm{d}s+\int_0^{\infty}G_2(t,s)h(s)\mathrm{d}s\\
&=\int_0^{+\infty}G(t,s)h(s)\mathrm{d}s,\end{aligned}$$

where $G(t,s)$, $G_1(t,s)$ and $G_2(t,s)$ are defined by (3.4), (3.5), (3.6) respectively. The proof is complete. □

Lemma 3.2. *The function $G_1(t,s)$ defined by* (3.5) *satisfies*

(i) *G_1 is a continuous function and $G_1(t,s)\ge 0$ for $(t,s)\in[0,+\infty)\times[0,+\infty)$;*
(ii) *$G_1(t,s)$ is strictly increasing in the first variable;*
(iii) *$G_1(t,s)$ is concave in the first variable for $0<s<t<+\infty$.*

Proof. It is easy to check that (i) holds.

Next, we prove that (ii) holds. For s fixed, we let

$$g_1(t) = \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1} - (t-s)^{\alpha-1}\right) \quad \text{for } s \le t,$$
$$g_2(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \quad \text{for } t \le s.$$

It is easy to check that $g_1(t)$ is strictly increasing on $[s, +\infty)$ and $g_2(t)$ is strictly increasing on $[0, s]$. Then we have the following cases:

Case 1: $t_1, t_2 \le s$ and $t_1 < t_2$. In this case, we have $g_2(t_1) < g_2(t_2)$, i.e. $G_1(t_1, s) < G_1(t_2, s)$.

Case 2: $s \le t_1, t_2$ and $t_1 < t_2$. In this case, we have $g_1(t_1) < g_1(t_2)$, i.e. $G_1(t_1, s) < G_1(t_2, s)$.

Case 3: $t_1 \le s \le t_2$ and $t_1 < t_2$. In this case, we have $g_2(t_1) \le g_2(s) = g_1(s) \le g_1(t_2)$. We claim that $g_2(t_1) < g_1(t_2)$. In fact, if $g_2(t_1) = g_1(t_2)$, then $g_2(t_1) = g_2(s) = g_1(s) = g_1(t_2)$, from the monotone of g_1 and g_2, we have $t_1 = s = t_2$, which contradicts with $t_1 < t_2$. This fact implies that $G_1(t_1, s) < G_1(t_2, s)$.

Finally, we prove that (iii) holds. For $0 < s < t < +\infty$, we have

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1} - (t-s)^{\alpha-1}\right).$$

Furthermore, we have

$$\frac{\partial^2 G_1(t, s)}{\partial t^2} = \frac{(\alpha-1)(\alpha-2)}{\Gamma(\alpha)}\left(t^{\alpha-3} - (t-s)^{\alpha-3}\right) < 0 \quad \text{for } 0 < s < t < +\infty.$$

Therefore, $G_1(t, s)$ is concave in the first variable for $0 < s < t < +\infty$. The proof is complete. □

Lemma 3.3. *For $k > 1$, then $G_1(t, s)$ defined by (3.5) has the following properties*

$$\min_{\frac{1}{k} \le t \le k} \frac{G_1(t, s)}{1 + t^{\alpha-1}} \ge \frac{1}{4k^2(1 + k^{\alpha-1})} \sup_{t \in [0, +\infty)} \frac{G_1(t, s)}{1 + t^{\alpha-1}}.$$

Proof. For s fixed, from the definition of $G_1(t, s)$, we know that the function $\frac{G_1(t,s)}{1+t^{\alpha-1}}$ achieves its maximum at $\xi \in [0, +\infty)$. Let

$$\eta = \inf\left\{\xi \in [0, +\infty) : \sup_{t \in [0, +\infty)} \frac{G_1(t, s)}{1 + t^{\alpha-1}} = \frac{G_1(\xi, s)}{1 + \xi^{\alpha-1}}\right\}.$$

So we divide the proof into three steps:

Step 1. If $\eta \le \frac{1}{k}$. From Lemma 3.2(ii), we have

$$\begin{aligned}
\min_{\frac{1}{k} \le t \le k} \frac{G_1(t, s)}{1 + t^{\alpha-1}} &\ge \frac{G_1\left(\frac{1}{k}, s\right)}{1 + k^{\alpha-1}} \ge \frac{G_1(\eta, s)}{1 + k^{\alpha-1}} = \frac{1 + \eta^{\alpha-1}}{1 + k^{\alpha-1}} \cdot \frac{G_1(\eta, s)}{1 + \eta^{\alpha-1}} \\
&\ge \frac{1}{1 + k^{\alpha-1}} \cdot \sup_{t \in [0, +\infty)} \frac{G_1(t, s)}{1 + t^{\alpha-1}} \\
&\ge \frac{1}{4k^2(1 + k^{\alpha-1})} \cdot \sup_{t \in [0, +\infty)} \frac{G_1(t, s)}{1 + t^{\alpha-1}}.
\end{aligned}$$

Step 2. If $\frac{1}{k} \le \eta \le k$. For $0 \le s \le \frac{1}{2k}$, from Lemma 3.2(iii), we have

$$\frac{G_1\left(\frac{1}{k}, s\right) - G_1\left(\frac{1}{2k}, s\right)}{\frac{1}{k} - \frac{1}{2k}} \ge \frac{G_1(\eta, s) - G_1\left(\frac{1}{2k}, s\right)}{\eta - \frac{1}{2k}},$$

i.e.

$$\left(\eta - \frac{1}{2k}\right) G_1\left(\frac{1}{k}, s\right) \ge \left(\eta - \frac{1}{k}\right) G_1\left(\frac{1}{2k}, s\right) + \frac{1}{2k} G_1(\eta, s) \ge \frac{1}{2k} G_1(\eta, s).$$

Therefore, we have

$$G_1\left(\frac{1}{k}, s\right) \ge \frac{1}{2\eta k} G_1(\eta, s). \tag{3.7}$$

From (3.7), we have

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} &\ge \frac{G_1\left(\frac{1}{k},s\right)}{1+k^{\alpha-1}} \ge \frac{G_1(\eta,s)}{2k\eta(1+k^{\alpha-1})} = \frac{1+\eta^{\alpha-1}}{2k^2(1+k^{\alpha-1})}\cdot\frac{G_1(\eta,s)}{1+\eta^{\alpha-1}}\\ &\ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.\end{aligned}$$

For $\frac{1}{2k}\le s\le\frac{1}{k}$, from Lemma 3.2(ii), we have

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} &\ge \frac{G_1\left(\frac{1}{k},s\right)}{1+k^{\alpha-1}} \ge \frac{G_1\left(\frac{1}{2k},s\right)}{1+k^{\alpha-1}} = \frac{G_1\left(\frac{1}{2k},s\right)}{G_1(\eta,s)}\cdot\frac{1+\eta^{\alpha-1}}{1+k^{\alpha-1}}\cdot\frac{G_1(\eta,s)}{1+\eta^{\alpha-1}}\\ &\ge \frac{1+\eta^{\alpha-1}}{(2k)^{\alpha-1}\eta^{\alpha-1}(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}\\ &\ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.\end{aligned}\tag{3.8}$$

For $\frac{1}{k}\le s\le\eta$, from Lemma 3.2(ii), we have

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} &\ge \frac{G_1\left(\frac{1}{k},s\right)}{1+k^{\alpha-1}} = \frac{G_1\left(\frac{1}{k},s\right)}{G_1(\eta,s)}\cdot\frac{1+\eta^{\alpha-1}}{1+k^{\alpha-1}}\cdot\frac{G_1(\eta,s)}{1+\eta^{\alpha-1}}\\ &\ge \frac{1+\eta^{\alpha-1}}{k^{\alpha-1}\eta^{\alpha-1}(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}\\ &\ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.\end{aligned}\tag{3.9}$$

For $\eta\le s$, from Lemma 3.2(ii), we have

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} &\ge \frac{G_1\left(\frac{1}{k},s\right)}{1+k^{\alpha-1}} = \frac{G_1\left(\frac{1}{k},s\right)}{G_1(\eta,s)}\cdot\frac{1+\eta^{\alpha-1}}{1+k^{\alpha-1}}\cdot\frac{G_1(\eta,s)}{1+\eta^{\alpha-1}}\\ &= \frac{1+\eta^{\alpha-1}}{k^{\alpha-1}\eta^{\alpha-1}(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}\\ &\ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.\end{aligned}\tag{3.10}$$

Step 3. If $k\le\eta$. For $0\le s\le\frac{1}{2k}$, from (3.7), we have

$$\begin{aligned}\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} &\ge \frac{G_1\left(\frac{1}{k},s\right)}{1+k^{\alpha-1}} \ge \frac{G_1(\eta,s)}{2k\eta(1+k^{\alpha-1})} = \frac{1+\eta^{\alpha-1}}{2k\eta(1+k^{\alpha-1})}\cdot\frac{G_1(\eta,s)}{1+\eta^{\alpha-1}}\\ &\ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.\end{aligned}$$

For $\frac{1}{2k}\le s\le\frac{1}{k}$, similar to (3.8), we have

$$\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} \ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.$$

For $\frac{1}{k}\le s\le\eta$, similar to (3.9), we have

$$\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} \ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.$$

For $\eta\le s$, similar to (3.10), we have

$$\min_{\frac{1}{k}\le t\le k}\frac{G_1(t,s)}{1+t^{\alpha-1}} \ge \frac{1}{4k^2(1+k^{\alpha-1})}\cdot\sup_{t\in[0,+\infty)}\frac{G_1(t,s)}{1+t^{\alpha-1}}.$$

The proof is complete. □

Remark 3.1. From the definition of $G_1(t,s)$, we have

$$\frac{G_1(t,s)}{1+t^{\alpha-1}} \le \frac{1}{\Gamma(\alpha)}, \qquad \frac{G(t,s)}{1+t^{\alpha-1}} \le L \quad \text{for } (t,s) \in [0,+\infty) \times [0,+\infty),$$

where $L = \frac{1}{\Gamma(\alpha)} + \frac{\sum_{i=1}^{m-2}\beta_i\xi_{m-2}^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)}$.

Remark 3.2. From the definition of $G_2(t,s)$, we have

$$\min_{\frac{1}{k}\le t\le k} \frac{G_2(t,s)}{1+t^{\alpha-1}} \ge \frac{1}{k^{2\alpha-2}(1+k^{\alpha-1})} \cdot \sup_{t\in[0,+\infty)} \frac{G_2(t,s)}{1+t^{\alpha-1}}.$$

Proof. In fact, from (3.6), we have

$$\begin{aligned}
\min_{\frac{1}{k}\le t\le k} \frac{G_2(t,s)}{1+t^{\alpha-1}} &= \min_{\frac{1}{k}\le t\le k} \frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)(1+t^{\alpha-1})} G_1(\xi_i,s) \\
&\ge \frac{\frac{1}{k^{\alpha-1}}}{1+k^{\alpha-1}} \frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}} G_1(\xi_i,s) \\
&\ge \frac{\frac{1}{k^{\alpha-1}}}{1+k^{\alpha-1}} \cdot \frac{1+\frac{1}{k^{\alpha-1}}}{k^{\alpha-1}} \sup_{t\in[0,+\infty)} \frac{\sum_{i=1}^{m-2}\beta_i t^{\alpha-1}}{\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)(1+t^{\alpha-1})} G_1(\xi_i,s) \\
&\ge \frac{1}{k^{2\alpha-2}(1+k^{\alpha-1})} \cdot \sup_{t\in[0,+\infty)} \frac{G_2(t,s)}{1+t^{\alpha-1}}. \qquad \square
\end{aligned}$$

Remark 3.3. For a fixed $k > 1$, let

$$\lambda(k) = \min\left\{\frac{1}{4k^2(1+k^{\alpha-1})}, \frac{1}{k^{2\alpha-2}(1+k^{\alpha-1})}\right\}.$$

From Lemma 3.3 and Remark 3.2, we have

$$\min_{\frac{1}{k}\le t\le k} \frac{G(t,s)}{1+t^{\alpha-1}} \ge \lambda(k) \sup_{0\le t<+\infty} \frac{G(t,s)}{1+t^{\alpha-1}}.$$

In this paper, we will use the following space E to study (1.1) and (1.2), which is denoted by

$$E = \left\{u \in C[0,+\infty) : \sup_{0\le t<+\infty} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty\right\}.$$

From [18], we know that E is a Banach space, equipped with the norm

$$\|u\| = \sup_{0\le t<+\infty} \frac{|u(t)|}{1+t^{\alpha-1}} < +\infty.$$

We definite the cone $P \subset E$ by

$$P = \{u \in E : u(t) \ge 0 \text{ on } [0,+\infty)\}.$$

Define an integral operator $T : P \to E$ by

$$Tu(t) = \int_0^{+\infty} G(t,s)a(s)f(u(s))\mathrm{d}s, \quad 0 \le t < +\infty, \tag{3.11}$$

where $G(t,s)$ defined by (3.4).

To obtain the complete continuity of T, the following lemma is still needed.

Lemma 3.4 ([18]). *Let $V = \{u \in E \mid \|u\| < l\}$ $(l > 0)$, $V_1 = \left\{\frac{u(t)}{1+t^{\alpha-1}} : u \in V\right\}$. If V_1 is equicontinuous on any compact intervals of $[0,+\infty)$ and equiconvergent at infinity, then V is relatively compact on E.*

Remark 3.4. V_1 is called equiconvergent at infinity if and only if for all $\epsilon > 0$, there exists $\nu(\epsilon) > 0$ such that for all $u \in V_1$, $t_1, t_2 \geq \nu$, it holds,

$$\left|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon.$$

Lemma 3.5. *Let* (C_1) *and* (C_2) *hold. Then* $T : P \to P$ *is completely continuous.*

Proof. First, it is easy to check that $T : P \to P$ is well-defined. From the definition of E, we can choose r_0 such that $\sup_{n\in N\setminus\{0\}} \|u_n\| < r_0$. Let $B_{r_0} = \sup\{f((1+t^{\alpha-1})u), u \in [0, r_0]\}$ and Ω be any bounded subset of P. Then the existence of $r > 0$ is such that $\|u\| \leq r$ for all $u \in \Omega$. Therefore, from Remark 3.1, we have

$$\begin{aligned}\|Tu\| &= \sup_{t\in[0,+\infty)} \frac{1}{1+t^{\alpha-1}} \left|\int_0^{+\infty} G(t,s)a(s)f(u(s))\mathrm{d}s\right| \\ &\leq LB_r \int_0^{+\infty} a(s)\mathrm{d}s < +\infty \quad \text{for } u \in \Omega.\end{aligned}$$

So $T\Omega$ is bounded. Moreover for any $T \in (0, +\infty)$ and $t_1, t_2 \in [0, T]$, without loss of generality, we may assume that $t_2 > t_1$. In fact,

$$\begin{aligned}\left|\frac{(Tu)(t_1)}{1+t_1^{\alpha-1}} - \frac{(Tu)(t_2)}{1+t_2^{\alpha-1}}\right| &\leq \int_0^{+\infty} \left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\quad + \frac{\sum_{i=1}^{m-2} \beta_i \xi_{m-2}^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} \left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_2^{\alpha-1}}\right| \int_0^{+\infty} a(s)f(u(s))\mathrm{d}s \\ &\leq \int_0^{+\infty} \left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\quad + \int_0^{+\infty} \left|\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\quad + \frac{\sum_{i=1}^{m-2} \beta_i \xi_{m-2}^{\alpha-1}}{\Gamma(\alpha) - \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1}} \left|\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_2^{\alpha-1}}\right| \int_0^{+\infty} a(s)f(u(s))\mathrm{d}s.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_0^{+\infty} \left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\leq \int_0^{t_1} \left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\quad + \int_{t_1}^{t_2} \left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\quad + \int_{t_2}^{+\infty} \left|\frac{G_1(t_1,s)}{1+t_1^{\alpha-1}} - \frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}\right| a(s)f(u(s))\mathrm{d}s \\ &\leq B_r \int_0^{t_1} \frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) + ((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})}{1+t_1^{\alpha-1}} a(s)\mathrm{d}s \\ &\quad + B_r \int_{t_1}^{t_2} \frac{(t_2^{\alpha-1} - t_1^{\alpha-1}) + (t_2-s)^{\alpha-1}}{1+t_1^{\alpha-1}} a(s)\mathrm{d}s + B_r \int_{t_2}^{+\infty} \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})}{1+t_1^{\alpha-1}} a(s)\mathrm{d}s \\ &\to 0 \quad \text{uniformly as } t_1 \to t_2. \end{aligned} \tag{3.12}$$

Similar to (3.12), we have

$$\int_0^{+\infty}\left|\frac{G_1(t_2,s)}{1+t_1^{\alpha-1}}-\frac{G_1(t_2,s)}{1+t_2^{\alpha-1}}\right|a(s)f(u(s))\mathrm{d}s\to 0\quad\text{uniformly as } t_1\to t_2. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\left|\frac{(Tu)(t_1)}{1+t_1^{\alpha-1}}-\frac{(Tu)(t_2)}{1+t_2^{\alpha-1}}\right|\to 0\quad\text{uniformly as } t_1\to t_2.$$

Hence, $T\Omega$ is locally equicontinuous on $[0,+\infty)$.

Next, we show that $T:E\to E$ is equiconvergent at ∞. For any $u\in\Omega$, we have

$$\int_0^{+\infty}a(s)f(u(s))\mathrm{d}s\le B_r\int_0^{+\infty}a(s)\mathrm{d}s<+\infty$$

and

$$\begin{aligned}\lim_{t\to+\infty}\left|\frac{(Tu)(t)}{1+t^{\alpha-1}}\right| &= \lim_{t\to+\infty}\frac{1}{1+t^{\alpha-1}}\int_0^{+\infty}G(t,s)a(s)f(u(s))\mathrm{d}s\\ &= \frac{\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}}{\Gamma(\alpha)\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)}\int_0^{+\infty}a(s)f(u(s))\mathrm{d}s\\ &\quad+\frac{\sum_{i=1}^{m-2}\beta_i}{\Gamma(\alpha)\left(\Gamma(\alpha)-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha-1}\right)}\int_0^{\xi_i}(\xi_i-s)^{\alpha-1}a(s)f(u(s))\mathrm{d}s<\infty.\end{aligned}$$

Hence, $T\Omega$ is equiconvergent at infinity.

Finally, we prove that T is continuous. Let $u_n\to u$ as $n\to+\infty$ in P. Since

$$\int_0^{+\infty}a(s)f(u(s))\mathrm{d}s<+\infty.$$

Then by the Lebesgue dominated convergence theorem and continuity of f, we can get

$$\int_0^{+\infty}a(s)f(u_n(s))\mathrm{d}s\to\int_0^{+\infty}a(s)f(u(s))\mathrm{d}s,\quad\text{as } n\to+\infty.$$

Therefore, by Remark 3.1, we have

$$\begin{aligned}\|Tu_n-Tu\| &\le L\left|\int_0^{+\infty}a(s)f(u_n(s))\mathrm{d}s-\int_0^{+\infty}a(s)f(u(s))\mathrm{d}s\right|\\ &\to 0,\quad\text{as } n\to+\infty.\end{aligned}$$

So T is continuous. By using Lemma 3.5, we obtain that $T:P\to P$ is completely continuous. □

4. Main results

For notational convenience, we denote by

$$M=L\int_0^{+\infty}a(s)\mathrm{d}s>0,\qquad m=\frac{\lambda(k)}{k^{\alpha-1}}\int_{\frac{1}{k}}^{k}a(s)\mathrm{d}s>0.$$

The main results of this paper are the following.

Theorem 4.1. *Suppose that conditions* (C_1)–(C_2) *hold. Let*

$$0<a<b<d\le c,$$

and suppose that f satisfies the following conditions:

(C_3) $f((1+t^{\alpha-1})u)<\frac{c}{M}$ *for all* $(t,u)\in[0,+\infty)\times[0,c]$;
(C_4) $f((1+t^{\alpha-1})u)>\frac{b}{m}$ *for all* $(t,u)\in\left[\frac{1}{k},k\right]\times[b,c]$;
(C_5) $f((1+t^{\alpha-1})u)<\frac{a}{M}$ *for all* $(t,u)\in[0,+\infty)\times[0,a]$.

Then fractional boundary value problem (1.1) and (1.2) has at least three positive solutions u_1 u_2 and u_3 satisfying

$$\|u_1\| < a, \qquad b < \gamma(u_2), \qquad a < \|u_3\| \quad \text{and} \quad \gamma(u_3) < b.$$

Proof. We define a nonnegative functional on E by $\gamma(u) = \min_{\frac{1}{k} \le t \le k} \frac{u(t)}{1+t^{\alpha-1}}$. We show that the conditions of Theorem 2.1 are satisfied.

Let $u \in \overline{P}_c$, then $\|u\| \le c$, that is

$$0 \le \frac{u(t)}{1+t^{\alpha-1}} \le c \quad \text{for } 0 \le t < +\infty.$$

Then assumption (C_3) implies

$$f(u) < \frac{c}{M} \quad \text{for all } (t, u) \in [0, +\infty) \times [0, c].$$

Therefore, from Remark 3.2 we have

$$\begin{aligned} \|Tu\| &= \sup_{0 \le t < +\infty} \frac{|Tu(t)|}{1+t^{\alpha-1}} \le \sup_{0 \le t < +\infty} \frac{1}{1+t^{\alpha-1}} \left| \int_0^{+\infty} G(t,s)a(s)f(u(s))\mathrm{d}s \right| \\ &< \frac{Lc}{M} \int_0^{+\infty} a(s)\mathrm{d}s = c. \end{aligned}$$

Hence, $T : P_c \to P_c$ and by Lemma 3.5, T is completely continuous.

Using an analogous argument, it follows from condition (C_5) that if $u \in \overline{P}_a$ then $\|Tu\| < a$. Condition (H_2) of Theorem 2.1 holds.

Let

$$u^*(t) = \frac{b+c}{2}(1+t^{\alpha-1}), \quad 0 \le t < +\infty.$$

It obvious that $u^*(t) \in P$ and $\|u^*\| = \frac{b+c}{2} < c$. From the definition of $\gamma(u)$, then

$$\gamma(u^*) = \frac{b+c}{2} > b.$$

Therefore we have

$$u^* \in \{x \in P(\gamma, b, d) : \gamma(x) > b\} \neq \varnothing.$$

On the other hand, for $u \in P(\gamma, b, d)$, we have

$$b \le \min_{\frac{1}{k} \le t \le k} \frac{u(t)}{1+t^{\alpha-1}} \le c.$$

Then assumption (C_4) implies

$$f(u) > \frac{b}{m} \quad \text{forall } (t, u) \in \left[\frac{1}{k}, k\right] \times [b, c].$$

Therefore from Remark 3.3 we have

$$\begin{aligned} \gamma(Tu) &= \min_{\frac{1}{k} \le t \le k} \frac{Tu(t)}{1+t^{\alpha-1}} \\ &= \min_{\frac{1}{k} \le t \le k} \frac{1}{1+t^{\alpha-1}} \int_0^{+\infty} G(t,s)a(s)f(u(s))\mathrm{d}s \\ &\ge \int_0^{+\infty} \min_{\frac{1}{k} \le t \le k} \frac{G(t,s)}{1+t^{\alpha-1}} a(s)f(u(s))\mathrm{d}s \\ &\ge \lambda(k) \int_{\frac{1}{k}}^{k} G\left(\frac{1}{k}, s\right) a(s)f(u(s))\mathrm{d}s \\ &> \frac{b}{m} \cdot \frac{\lambda(k)}{k^{\alpha-1}} \int_{\frac{1}{k}}^{k} a(s)\mathrm{d}s \\ &= b. \end{aligned}$$

That is for all $u \in P(\gamma, b, d)$, $\gamma(Tu) > b$. Condition (H_1) of Theorem 2.1 holds.

Finally, if $u \in P(\gamma, b, c)$ with $\gamma(Tu) > d$ then $\|u\| \le c$ and $b \le \frac{u(t)}{1+t^{\alpha-1}} \le c$ and from assumption (C_4) we can show $\gamma(Tu) > b$. Condition (H_3) of Theorem 2.1 holds.

As a consequence of Theorem 2.1, T has at least three fixed points u_1, u_2, u_3 such that $\|u_1\| < a$, $b < \gamma(u_2)$, $a < \|u_3\|$ with $\gamma(u_3) < b$. These fixed points are solutions of (1.1)-(1.2) and the proof is complete. □

5. Example

Example 5.1. We consider the following m-point fractional boundary value problem on an infinite interval

$$D_{0+}^{\frac{5}{2}}u(t) + \mathrm{e}^{-t}f(t, u(t)) = 0, \quad 0 < t < +\infty, \tag{5.1}$$

$$u(0) = u'(0) = 0, \qquad D^{\frac{3}{2}}u(+\infty) = \frac{1}{4}u\left(\frac{1}{4}\right) + \frac{1}{2}u(1), \tag{5.2}$$

where

$$f(t,u) = \begin{cases} \dfrac{1}{1000}|\sin t| + 10\left(\dfrac{u}{1+t^{\frac{3}{2}}}\right)^9, & u \le \dfrac{1}{2}, \\ \dfrac{1}{1000}|\sin t| + 10\left(\dfrac{u}{1+t^{\frac{3}{2}}}\right)^9 + 10^5\left(u - \dfrac{1}{2}\right), & \dfrac{1}{2} \le u \le 1, \\ \dfrac{1}{1000}|\sin t| + 10\left(\dfrac{u}{1+t^{\frac{3}{2}}}\right)^9 + 5\times 10^4, & 1 \le u. \end{cases}$$

In this case, $\alpha = \frac{5}{2}$, $a(t) = \mathrm{e}^{-t}$, $\beta_1 = \frac{1}{4}$, $\beta_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{4}$, $\xi_2 = 1$. Select $a = \frac{1}{2}$, $b = 1$, $c = 10^5$ and $k = 4$. Then by direct calculation we can obtain that

$$L = \frac{4}{3\sqrt{\pi}} + \frac{2}{3\sqrt{\pi}\left(\frac{3}{4}\sqrt{\pi} - \frac{17}{32}\right)} \approx 1.226, \qquad \lambda(4) = \frac{1}{576},$$

$$\int_0^{+\infty} \mathrm{e}^{-t}\mathrm{d}t = 1, \qquad \int_{\frac{1}{4}}^{4} \mathrm{e}^{-t}\mathrm{d}t = \mathrm{e}^{-\frac{1}{4}} - \mathrm{e}^{-4} \approx 0.76,$$

$$M = L \approx 1.226, \qquad m \approx 0.000165.$$

So the nonlinear term f satisfies

$$f(t, (1+t^{\frac{3}{2}})u) < 0.002 + 5\times 10^4 < 81566.01 \approx \frac{c}{M}, \quad (t,u) \in [0,+\infty) \times [0, 10^5],$$

$$f(t, (1+t^{\frac{3}{2}})u) > 6250 > 6060.6061 \approx \frac{b}{m}, \quad (t,u) \in \left[\frac{1}{4}, 4\right] \times [1, 10^5],$$

$$f(t, (1+t^{\frac{3}{2}})u) < 0.001 + 10\left(\frac{1}{2}\right)^9 < 0.40783 \approx \frac{a}{M}, \quad (t,u) \in [0,+\infty) \times \left[0, \frac{1}{2}\right].$$

Then by an application of Theorem 4.1 the fractional boundary value problem (5.1) and (5.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\sup_{0\le t<+\infty} \frac{|u_1(t)|}{1+t^{\alpha-1}} < \frac{1}{2}, \qquad 1 < \min_{\frac{1}{4}\le t\le 4} \frac{u_2(t)}{1+t^{\alpha-1}},$$

$$\frac{1}{2} < \sup_{0\le t<+\infty} \frac{|u_3(t)|}{1+t^{\alpha-1}} \quad \textit{with} \quad \sup_{0\le t<+\infty} \frac{|u_3(t)|}{1+t^{\alpha-1}} < 1.$$

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References

[1] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.
[2] I. Podlubny, Fractional Differential Equations, in: Mathematics in Sciences and Engineering, vol. 198, Academic Press, San Diego, 1999.
[3] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B. V., Amsterdam, 2006.
[4] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, Cambridge Academic, Cambridge, UK, 2009.

[5] C. Bai, Positive solutions for nonlinear fractional differential equations with coefficient that changes sign, Nonlinear Anal. 64 (2006) 677–685.
[6] C. Bai, Triple positive solutions for a boundary value problem of nonlinear fractional differential equation, Electron. J. Qual. Theory Differ. Equ. 24 (2008) 1–10.
[7] Z. Bai, H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, J. Math. Anal. Appl. 311 (2005) 495–505.
[8] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2008) 1340–1350.
[9] A.M.A. El-Sayed, A.E.M. El-Mesiry, H.A.A. El-Saka, On the fractional-order logistic equation, Appl. Math. Lett. 20 (2007) 817–823.
[10] M. El-Shahed, Positive solutions for boundary value problem of nonlinear fractional differential equation, Abstr. Appl. Anal. 2007 (2007) doi:10.1155/2007/10368. Article ID 10368, 8 pages.
[11] M.Q Feng, X.M. Zhang, W.G. Ge, New existence results for higher-order nonlinear fractional differential equation with integral boundary conditions, Bound. Value Probl. 2011 (2011) 1–20.
[12] C.F. Li, X.N. Luo, Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations, Comput. Math. Appl. 59 (2010) 1363–1375.
[13] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008) 2677–2682.
[14] V. Lakshmikantham, A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Appl. Math. Lett. 21 (2008) 828–834.
[15] S. Liang, J.H. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equation, Nonlinear Anal. 71 (2009) 5545–5550.
[16] X.K. Zhang, W.G. Ge, Unbounded solutions for a fractional boundary value problem on the infinite interval, Acta Appl. Math. 109 (2010) 495–505.
[17] R.W. Leggett, L.R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana Univ. Math. J. 28 (1979) 673–688.
[18] Y.S. Liu, Existence and unboundedness of positive solutions for singular boundary value problems on half-line, Appl. Math. Comput. 1404 (2003) 543–556.